Nonlinear diffusions with rough weights: sharp Widder theory.

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Aronson (SIAM J. Math. Anal. 1981), settled the higher-dimensional case and studied evolutions driven by general linear divergence-form parabolic equations.

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In fact, this is clear from the explicit representation, again due to Widder for general, nonnegative solutions:

$$u(t,x) = ct^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)\,\mathrm{d}y,$$

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Positivity here is essential. Sign-changing solutions can have whatever growth at infinity, as shown by Tychonov (Mat. Sb., 1935).

In fact, the representation formula above is general. One can show that if u is a nonnegative classical solution of the heat equation, there exists a nonnegative Borel measure μ such that

$$u(,t) = p_t * \mu \text{ for all } t > 0,$$

where

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Conversely, if μ satisfies the above bound, $u(,t) = p_t * \mu$ is a solution to the heat equation with initial datum μ in the sense of measures.

Classical results for the PME

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$$\mu(B_R) = \mathcal{O}\left(R^{N+\frac{2}{m-1}}\right) \qquad \text{as } R \to +\infty\,. \tag{2}$$

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The growth estimate (2) also holds for solutions at any given positive time, and this can be used to prove uniqueness for general non-negative solutions taking the same initial trace.

B.E.J. Dahlberg and C.E. Kenig (Commun. PDE 1984) proved uniqueness for general positive solutions, without any a priori growth. They use continuity of solutions, later proved by them (TAMS 1993).

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2. *Prove that if* $u \geq 0$ *is a very weak solution* $\implies \exists!\ \mu \in X$ *an initial trace of* u*, and* $u \in Y$. This is contained in the works of Aronsson-Caffarelli and Dahlberg-Kenig for the PME.

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2. *Prove that if $u \geq 0$ is a very weak solution $\implies \exists!\ \mu \in X$ an initial trace of u, and $u \in Y$*. This is contained in the works of Aronsson-Caffarelli and Dahlberg-Kenig for the PME.
3. *General non-negative solutions taking the same initial trace are equal*. This was proven by Dahlberg-Kenig for the PME.

Some further comments

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  Significant generalizations have been proved in Gonzálvez, F. Quirós, F. Soria, Z. Vondraček (arXiv 2025), always in the linear case, where in particular fractional-type operators with possibly rough kernels are considered, which are the natural counterparts of elliptic operators with measurable coefficients in the local case.

The porous medium equation with rough weights

We investigate initial traces, *a priori* estimates, existence, and uniqueness for solutions to the following weighted porous medium equation (WPME):

$$\rho(x)\, u_t = \Delta(u^m) \qquad \text{in } \ \mathbb{R}^N \times (0,T)\,, \tag{3}$$

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The weight ρ satisfies

$$\begin{gathered} \underline{C}\,(1+|x|)^{-\gamma} \leq \rho(x) \leq \overline{C}\,|x|^{-\gamma} \qquad \text{for a.e. } x \in \mathbb{R}^N\,, \\ \text{for some } \gamma \in [0,2) \text{ and constants } \overline{C} > \underline{C} > 0\,. \end{gathered} \tag{4}$$

On the assumptions on the weight

- Our results are new *even when* ρ is bounded and bounded away from zero, i.e. $\gamma = 0$ in (4).

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- The WPME is connected to the PME on a Riemannian manifold with non-positive curvature, see Vázquez (JMPA 2015), Grillo, Muratori, Vazquez (Adv. Math. 2017, Math. Ann. 2019).

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- The WPME is connected to the PME on a Riemannian manifold with non-positive curvature, see Vázquez (JMPA 2015), Grillo, Muratori, Vazquez (Adv. Math. 2017, Math. Ann. 2019). $\gamma \in (0, 2)$ corresponds to manifolds with negative curvature decaying at spatial infinity, $\gamma = 2$ corresponds to the hyperbolic space. The case $\gamma > 2$ is related to superexponential curvature behaviour, in which case uniqueness does not hold even for bounded solutions.

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- As concerns existence of solutions in the right class, we miss the Aronson–Bénilan inequality, namely

$$\Delta u^{m-1} \geq -\frac{C}{t} \quad \text{in the distributional sense.}$$

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- Continuity of solutions is not available. This was fundamental both for the Aronson-Caffarelli proof of existence of initial traces, and for the Dahlberg and Kenig results on uniqueness. We replace this property by local boundedness of solutions (to be proved).
- Other tools, like the scale invariance of the equation and the validity of Aleksandrov's reflection principle, are replaced here by potential methods

On the concept of solution

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Definition 1

We say that a function

$$u \in L^1_{\rho,\mathrm{loc}}\big(\mathbb{R}^N \times (0,T)\big) \cap L^m_{\mathrm{loc}}\big(\mathbb{R}^N \times (0,T)\big)$$

is a very weak solution to (3) *if*

$$\int_0^T \int_{\mathbb{R}^N} (u\,\psi_t\,\rho + u^m\,\Delta\psi)\,dxdt = 0 \tag{5}$$

for all $\psi \in C_c^\infty\big(\mathbb{R}^N \times (0,T)\big)$.

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Let ρ satisfying (4). *Let u be a non-negative solution to* (3), *in the sense of Definition 1. Then there exists a non-negative Radon measure μ s.t.:*

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\varphi(x)\,u(x,t)\,\rho(x)\,dx=\int_{\mathbb{R}^N}\varphi\,d\mu\qquad\forall\varphi\in C_c\big(\mathbb{R}^N\big)\,,$$

where $t\mapsto\int_{\mathbb{R}^N}\varphi(x)\,u(x,t)\,\rho(x)\,dx$ has a continuous version in $(0,T)$. Furthermore:

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where $t\mapsto\int_{\mathbb{R}^N}\varphi(x)\,u(x,t)\,\rho(x)\,dx$ *has a continuous version in* $(0,T)$. *Furthermore:*

$$\mu(B_R)\leq C\left[t^{-\frac{1}{m-1}}R^{N-\gamma+\frac{2-\gamma}{m-1}}+t^{\frac{N-\gamma}{2-\gamma}}\left(\fint_{B_\varepsilon\times(t,t+\delta)}u\,\rho\,dxds\right)^{1+\frac{N-\gamma}{2-\gamma}(m-1)}\right]$$

for all $\varepsilon\in(0,1)$, *all* $t\in(0,T)$, *and all* $\delta\in(0,T-t)$.

It has been shown by Muratori and Petitt that there exists solutions behaving like

$$U(x,t) = |x|^{\frac{2-\gamma}{m-1}} (T-t)^{-\frac{1}{m-1}},$$

when $|x| \to +\infty$ and $t \to T$, where T is any given, fixed positive constant. Such functions can be used to show sharpness of the time and space behaviour predicted by the above Theorem.

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To introduce our next results, let us make the following definition:

Definition 2

Let $r \geq 1$, μ a Radon measure. Define

$$\|\mu\|_{1,r} = \sup_{R \geq r} R^{-(N-\gamma)-\frac{2-\gamma}{m-1}} \, |\mu|(B_R) \, ,$$

where $|\mu|$ is the total variation of μ.

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where $|\mu|$ is the total variation of μ. We denote by X the Banach space of all Radon measures μ in $\mathbb{R}^N$ such that $\|\mu\|_{1,r} < +\infty$. By the local finiteness of the total variation of μ, such a definition is independent of $r \geq 1$ and all norms $\|\cdot\|_{1,r}$ are equivalent.

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$$\ell(\mu) := \lim_{r \to +\infty} \|\mu\|_{1,r} \, .$$

We say that $\mu \in X_0$ if $\ell(\mu) = 0$. Finally, if $f \in L^\infty_{\rm loc}(\mathbb{R}^N)$, we set $\|f\|_{\infty,r} = \sup_{R \geq r} R^{-\frac{2-\gamma}{m-1}} \, \|f\|_{L^\infty(B_R)}$.

We also let $\Phi_\alpha(x) = \left(1 + |x|^2\right)^{-\alpha}$, $\alpha > \frac{2-\gamma}{2(m-1)} + \frac{N-\gamma}{2}$. Than X is continuously embedded in $L^1(\Phi_\alpha \rho \, dx)$.

Theorem 2 (Existence of solutions with measure initial data)

Let $\mu \in X$ *be a Radon measure. Then there exists a solution* u *of* (3) *with initial datum* μ*, with* $T = \mathsf{T}(\mu)$ *given by*

$$\mathsf{T}(\mu) = \frac{\mathsf{C}_1}{[\ell(\mu)]^{m-1}} \quad \textit{if} \;\; \mu \in X \setminus X_0 \, , \qquad \mathsf{T}(\mu) = +\infty \quad \textit{if} \;\; \mu \in X_0 \, , \tag{6}$$

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Moreover, the solution u satisfies

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Moreover, the solution u *satisfies*

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Furthermore, $\|u(t)\|_{\infty,r} \le \mathsf{C}_3\, t^{-\lambda}\, \|\mu\|_{1,r}^{\theta\lambda} \;\; \forall t \in (0, \mathsf{T}_r(\mu))$ *, for a suitable* T_r*, the comparison principle holds, and if* $\mu \in X_0$*, then* $u \in W^{1,\infty}_{\mathrm{loc}}((0, \mathsf{T}(\mu))\,; X_0)$ *and*

$$\operatorname*{ess\,lim}_{|x|\to+\infty} |x|^{-\frac{2-\gamma}{m-1}}\, u(x,t) = 0 \qquad \forall t > 0\,. \tag{8}$$

Uniqueness

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Theorem 3 (Uniqueness without global assumptions)

Let u and v be non-negative solutions to (3)*, in the sense of Definition 1, such that:*

$$\operatorname*{ess\,lim}_{t\to 0} \int_{\mathbb{R}^N} [u(x,t)-v(x,t)]\,\varphi(x)\,\rho(x)\,dx = 0 \qquad \forall\varphi\in C_c\big(\mathbb{R}^N\big)\,. \tag{9}$$

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Then $u = v$.

As a consequence, as far as non-negative solutions are concerned, the initial datum problem associated to (3) is solvable if and only if $\mu \in X$, and in that case it admits exactly one non-negative solution. Moreover, for all $t \in (0, T)$ we have

$$\|u(t)\|_{L^\infty(B_R)} = \mathcal{O}\left(R^{\frac{2-\gamma}{m-1}}\right) \qquad \text{as } \ R \to +\infty\,.$$

In order to prove the uniqueness result, we first obtain quantitative uniform estimates for locally bounded solutions, which might be defined only in space-time sets of the form $\Omega \times (\tau_1, \tau_2)$, where $\Omega \subset \mathbb{R}^N$ is a (possibly unbounded) domain and $0 < \tau_1 < \tau_2$.

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In the unweighted framework Dahlberg-Kenig even proved that non-negative very weak solutions are space-time continuous. In our case however, the inclusion of a (possibly rough) weight ρ precludes the possibility of applying Hölder-regularity results therefore we need to replace continuity with local boundedness, which turns out to be sufficient for our purposes.

Theorem 4

Let $\Omega \subset \mathbb{R}^N$, $0 \leq \tau_1 < \tau_2$. *Let* u *be a very weak, locally bounded, local very weak solution to* (3) *in* $\Omega \times (\tau_1, \tau_2)$.

Theorem 4

Let $\Omega \subset \mathbb{R}^N$, $0 \leq \tau_1 < \tau_2$. *Let* u *be a very weak, locally bounded, local very weak solution to* (3) *in* $\Omega \times (\tau_1, \tau_2)$. *Let* $R \geq 1$ *be such that* $B_{2R} \Subset \Omega$, *and for* $\tau_1 < t_1 < t_2 < T^* < \tau_2$ *consider the pair of cylinders*

$$\overline{Q} = B_{2R} \times (t_1, T^*) , \qquad \underline{Q} = B_R \times (t_2, T^*) .$$

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Let $\Omega \subset \mathbb{R}^N$, $0 \le \tau_1 < \tau_2$. *Let* u *be a very weak, locally bounded, local very weak solution to* (3) *in* $\Omega \times (\tau_1, \tau_2)$. *Let* $R \ge 1$ *be such that* $B_{2R} \Subset \Omega$, *and for* $\tau_1 < t_1 < t_2 < T^* < \tau_2$ *consider the pair of cylinders*

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Then, for all $\varepsilon \in (0, \varepsilon_0)$, *and* $m \in (1,2)$, *and suitable* $\alpha, \beta > 0$:

$$R^{-\alpha} \|u\|_{L^\infty(\underline{Q})} \le C_1 \left[\left(R^{-\beta} \|u\|_{L^1_\rho(\overline{Q})} \right)^{\frac{1}{2-m}} + \left(\frac{1}{t_2 - t_1} \right)^{\frac{1}{m-1}} \right]$$

whereas, for any $m > 1$:

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where $\mathsf{S} = \sup_{t \in (t_1, T^*)} \int_{B_{2R}} |u(x,t)|\, \rho(x)\, dx$.

(Almost) optimality

By using again the "solutions"

$$U(x,t) = |x|^{\frac{2-\gamma}{m-1}} (T-t)^{-\frac{1}{m-1}} ,$$

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Besides, (almost) time optimality can also be seen using the friendly giant V,

$$V(x,t) = W(x)\,t^{-\frac{1}{m-1}} \qquad \text{for } \ (x,t) \in \Omega \times (0,+\infty)\,,$$

for $\Omega \subset \mathbb{R}^N$ bounded, where $W \geq 0$ satisfies $-\Delta W^m = \rho\,W$, with Dirichlet b.c..

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Theorem 5 (Local boundedness)

Let $\Omega \subset \mathbb{R}^N$ *be a (possibly unbounded) domain and* $0 \leq \tau_1 < \tau_2$*. Let* u *be a non-negative local very weak solution to* (3)*. Then* $u \in L^\infty_{\mathrm{loc}}(\Omega \times (\tau_1, \tau_2))$*.*

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We stress that such a result is crucially exploited both in the proof of Aronson-Caffarelli inequality and in the result on uniqueness, as it ensures that non-negative solutions can always be approximated (locally) by sequences of more regular solutions.

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This has further consequences, namely:

Corollary (Local smoothing estimates for non-negative solutions)

Let u be a nonnegative local very weak solution. Then the smoothing estimates of Theorem (4) *hold.*

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Corollary (Very weak solutions are strong energy)

Let $\Omega \subset \mathbb{R}^N$ be a (possibly unbounded) domain and $0 \le \tau_1 < \tau_2$. Let u be either a locally bounded or a non-negative local very weak solution. Then u is a strong energy solution. In particular,

$$u^m \in H^1_{\mathrm{loc}}\big((\tau_1,\tau_2); L^2_{\rho,\mathrm{loc}}(\Omega)\big) \cap L^2_{\mathrm{loc}}\big((\tau_1,\tau_2); H^1_{\mathrm{loc}}(\Omega)\big)\,,$$

THANKS FOR YOUR ATTENTION!